Rigid Body Motion

A Comprehensive Introduction

Motivation

Motiviation

- Rigid Body Motion is all around us
- Rigid Body Motion is **the** game physics concept in use
- Bullet, Havok, ODE all use this concept

You need to know about it if you want to work with and understand physics engines!

Overview

1. Short Math Primer
    1. Vector
    2. Dot Product
    3. Cross Product
    4. Calculus
2. Kinematics
    1. Linear Motion
    2. Angular Motion in 2D
3. Statics
    1. Force
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    4. Field Forces
    5. Contact Forces
    6. Moment
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    8. Center of Mass
4. Kinetics
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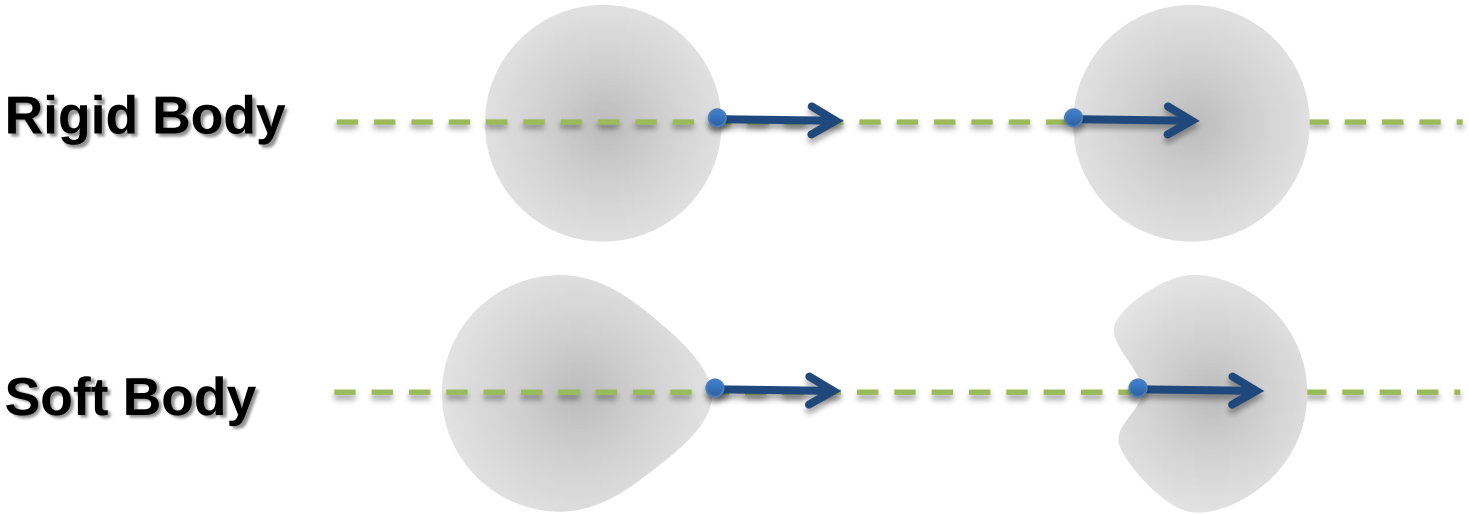
    2. Angular Momentum
    3. Integration
    4. Inertia Tensor in 2D
    5. Inertia Tensor in 3D
5. Bibliography

Rigid Body

Definition

A **rigid body (Starrkörper)** is a solid body that is not deformable.



Game Engine Example

Rigid Body Update Code

```
for each (RigidBody* body in scene->rigidBodies) {
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Where We Are

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```

Short Math Primer

Vector

Definition

A vector is a matrix with a single column:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

v or $\|\mathbf{v}\|$ denotes the magnitude of the vector.

This is just a stub to show the notation : -)

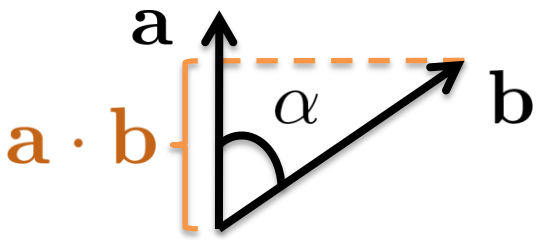
You all know everything about vertices and matrices from your Linear Algebra lecture.

Dot Product

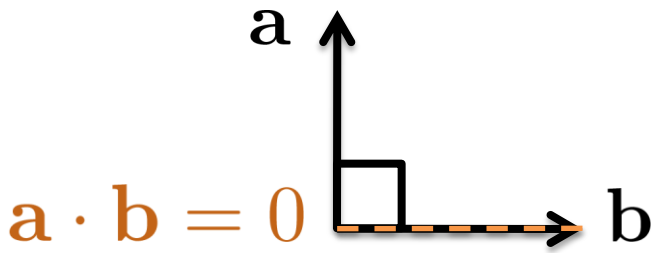
Definition

The **dot product (Skalarproduct)** of two vectors $\mathbf{a} \cdot \mathbf{b}$ is the scalar $a\,b\cos\alpha$, where α is the angle between the two vectors.

$a = 1$



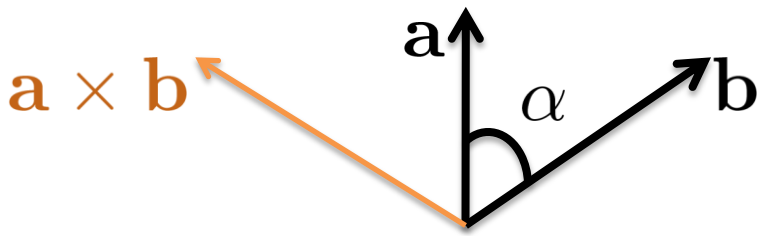
$a \perp b$



Cross Product

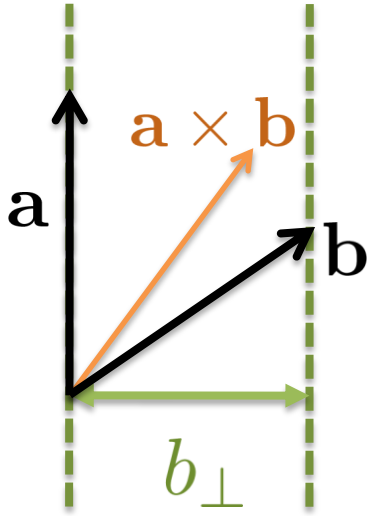
Definition

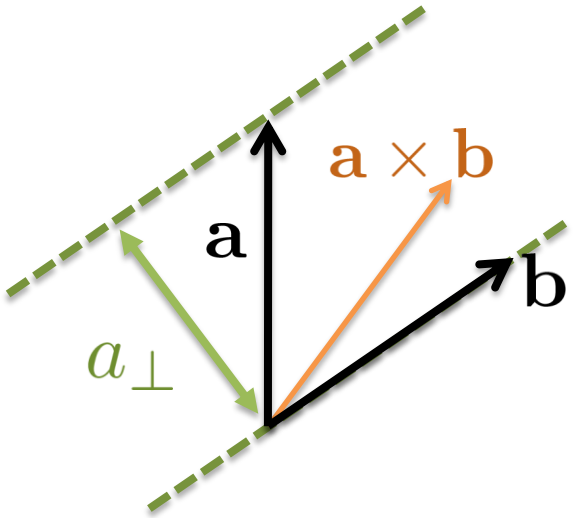
The **cross product (Kreuzprodukt)** of two vectors $\mathbf{a} \times \mathbf{b}$ is a vector that is **perpendicular** to both $\mathbf{a}$ and $\mathbf{b}$. The length of the resulting vector is $a\,b\sin\alpha$, where α is the angle between $\mathbf{a}$ and $\mathbf{b}$.



You can use the right-hand rule (Rechte-Hand-Regel) to visualize the cross product.

Cross Product





$$\|\mathbf{a} \times \mathbf{b}\| = a\, b_\perp \qquad \|\mathbf{a} \times \mathbf{b}\| = a_\perp\, b$$

$$\mathbf{a} \perp \mathbf{b} \implies \|\mathbf{a} \times \mathbf{b}\| = a\, b$$

Branches of Mechanics

There are **three** branches of mechanics that are of interest to us:

Definition

Kinematics (Kinematik) describes the motion of objects without regard for the reason of their movement.

(from Wikipedia)

Definition

Dynamics (Dynamik) or also called **Kinetics (Kinetik)** examines the correlation between an object‘s motion and the reasons for it.

(from Wikipedia)

Definition

Statics (Statik) analyzes the motion of objects without regard for the reason of their movement.

(from Wikipedia)

Kinematics

Linear Motion

Let's look at the movement of a simple **particle (Massenpunkt)** .
We identify a particle i by its **position** $\mathbf{r_i}$.

$$\mathbf{r_i} = \begin{pmatrix} r_{ix} \\ r_{iy} \\ r_{iz} \end{pmatrix}$$

The **velocity** $\mathbf{v_i}$ of a particle i is its change of position over time:

$$\mathbf{v_i} = \frac{d\mathbf{r_i}}{dt} = \dot{\mathbf{r}}_\mathbf{i}$$

The **speed (Tempo)** of a particle is simply the magnitude of its velocity: v_i .

Likewise the **acceleration** $\mathbf{a_i}$ is the change of velocity over time:

$$\mathbf{a_i} = \frac{d\mathbf{v_i}}{dt} = \dot{\mathbf{v}}_\mathbf{i} = \ddot{\mathbf{r}}_\mathbf{i}$$

Angular Motion in 2D

Imagine that a particle moves on a circle around the origin.

We can exactly describe the particle's position by the angle ϕ between the particle and the x-axis at any moment of time.

Like with **linear motion**, we can examine the change of ϕ over time and get the **angular velocity** ω:

$$\omega = \frac{d\phi}{dt} = \dot{\phi}$$

Similarly the **angular acceleration** α is the change of ω over time:

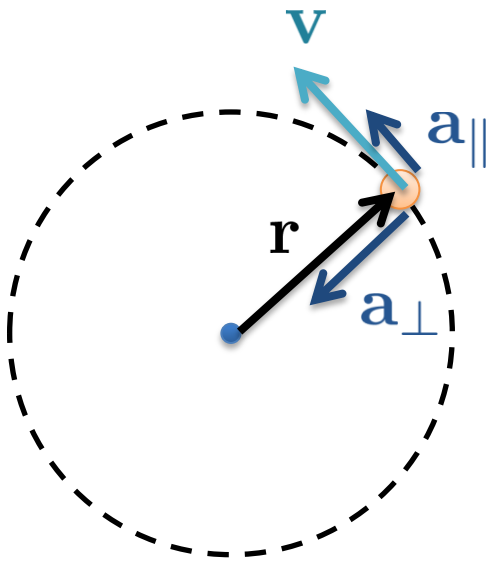
$$\alpha = \frac{d\omega}{dt} = \dot{\omega} = \ddot{\phi}$$

Example: Circular Movement

$$\mathbf{r} = r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

$$\Rightarrow \mathbf{v} = \dot{\mathbf{r}} = r \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \cdot \dot{\phi} = r\,\omega \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \implies v = r\,\omega$$

And $\mathbf{v} \perp \mathbf{r}$ because $\mathbf{v} \cdot \mathbf{r} = r^2\,\omega\,(-\sin\phi\,\cos\phi + \sin\phi\,\cos\phi) = 0$



Example: Circular Movement in 3D

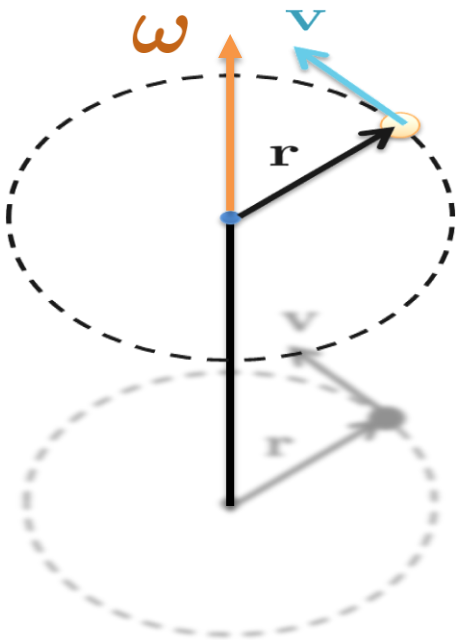
If we look at the same rotation in 3D nothing changes.
We give $\boldsymbol{\omega}$ a direction as rotation axis and set:

$$\|\boldsymbol{\omega}\| = \omega$$

Since $\omega \perp \mathbf{v}$ it follows that:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

with $v = \omega r$

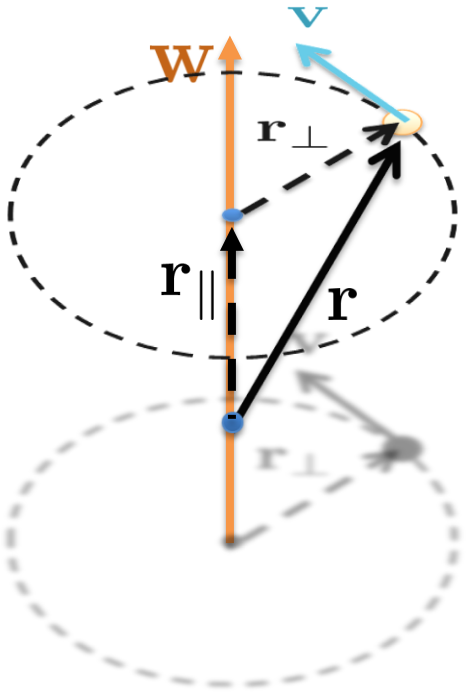


Example: Rotation around an Axis in 3D

In the general case we have $\omega \not\perp \mathbf{v}$, so does our equation fail?

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

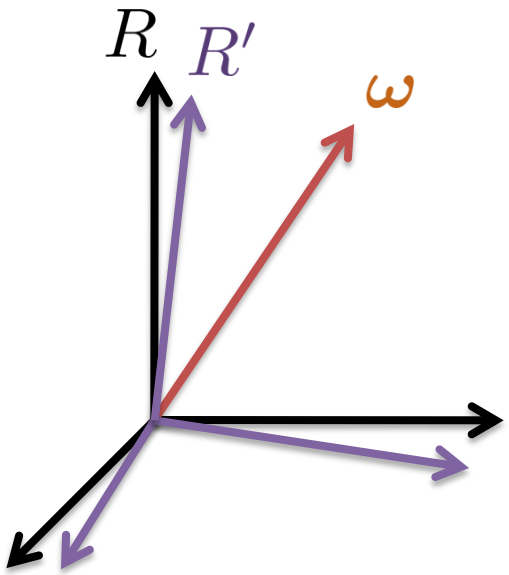
with $v = \omega\, r_\perp$



Angular Motion in 3D

In 3D we can't use a single angle to describe the orientation. Instead we use a rotation matrix R.

The derivative of a matrix is again a matrix, so instead we need something simpler:
The change of the rotation matrix from one moment to the next is nothing else but a rotation itself.
We can describe any rotation using the rotation axis ω with its magnitude being the amount of rotation.



Angular Motion in 3D

It can be shown that $\dot{R} = \omega \times R$ (with a column-wise cross product).

Now we know how to get from $\boldsymbol{\omega}$ to $\dot{R}$ and can use the formulas that we have deduced when rotating a point around an axis:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

to convert from angular velocity to linear velocity.

Summary

Linear Motion:

- Position $\mathbf{r}$
- Velocity $\mathbf{v}$
- Acceleration $\mathbf{a}$

Angular Motion:

- Orientation ϕ or R
- Angular Velocity ω
  - $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$
- Angular Acceleration α

Statics

Force

Definition

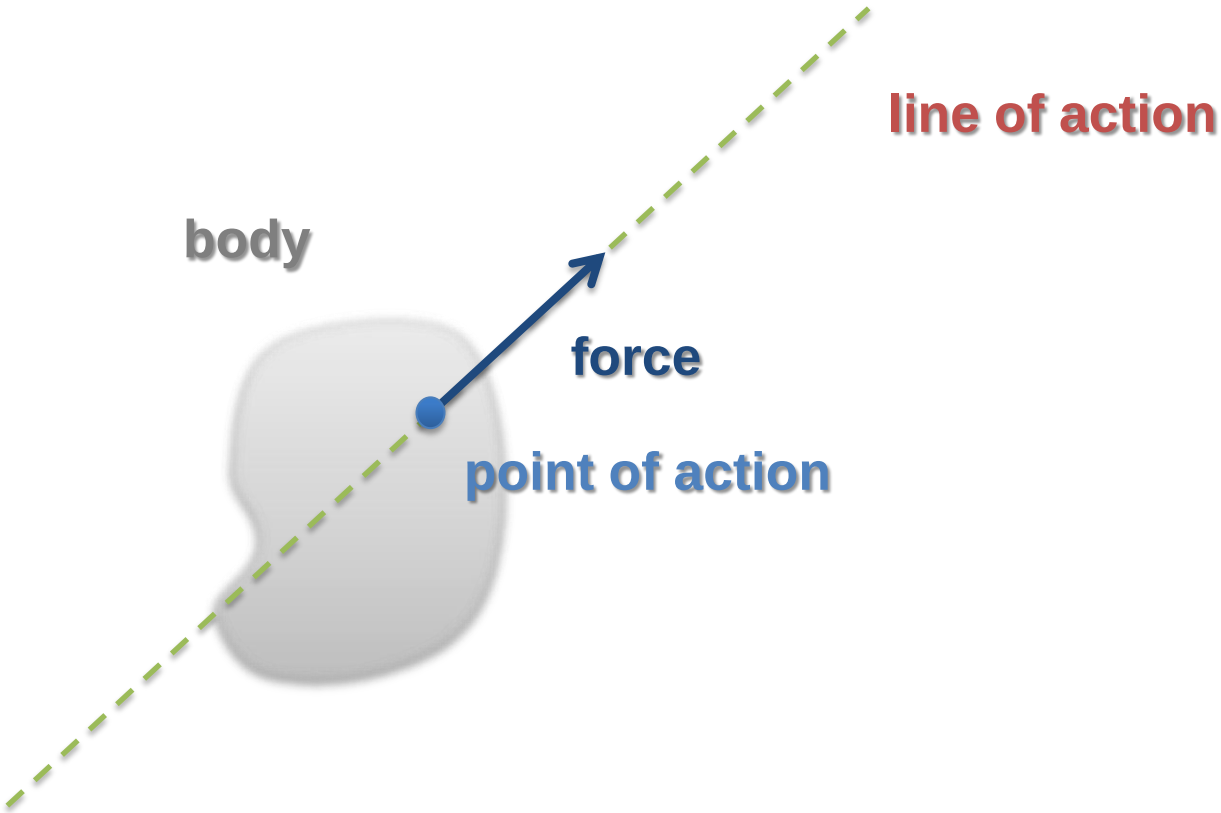
A **force (Kraft)** is any influence that causes a free body to undergo an acceleration.

A force has both **magnitude** and **direction**, making it a vector quantity.

(from Wikipedia)

It also has a **point of action (Angriffspunkt)**: the point where the force is applied; and a **line of action (Wirkungslinie)**: the line along the force's direction through the point of action.

Force



Newton's Laws of Motion

Second Law

The acceleration of a body is proportional to the net force acting on the body and this acceleration is in the same direction as the net force.

$$\mathbf{F} = m\,\mathbf{a}$$

Third Law

For every force acting on a body, there is an equal and opposite reacting force.

$$actio = -reactio$$

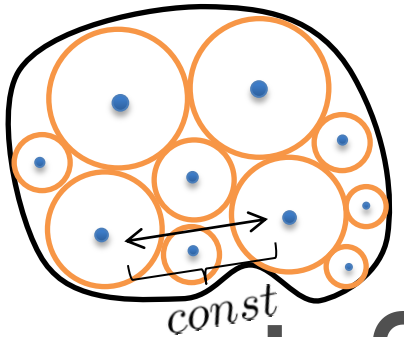
Rigid Bodies

Definition

A rigid body (Starrkörper) is a solid body that is not deformable.

If we take a body to be made-up of particles, then the definition means that the distance between any two particles always remains constant.

This, of course, is an idealization, because every body can be deformed if enough stress is applied.

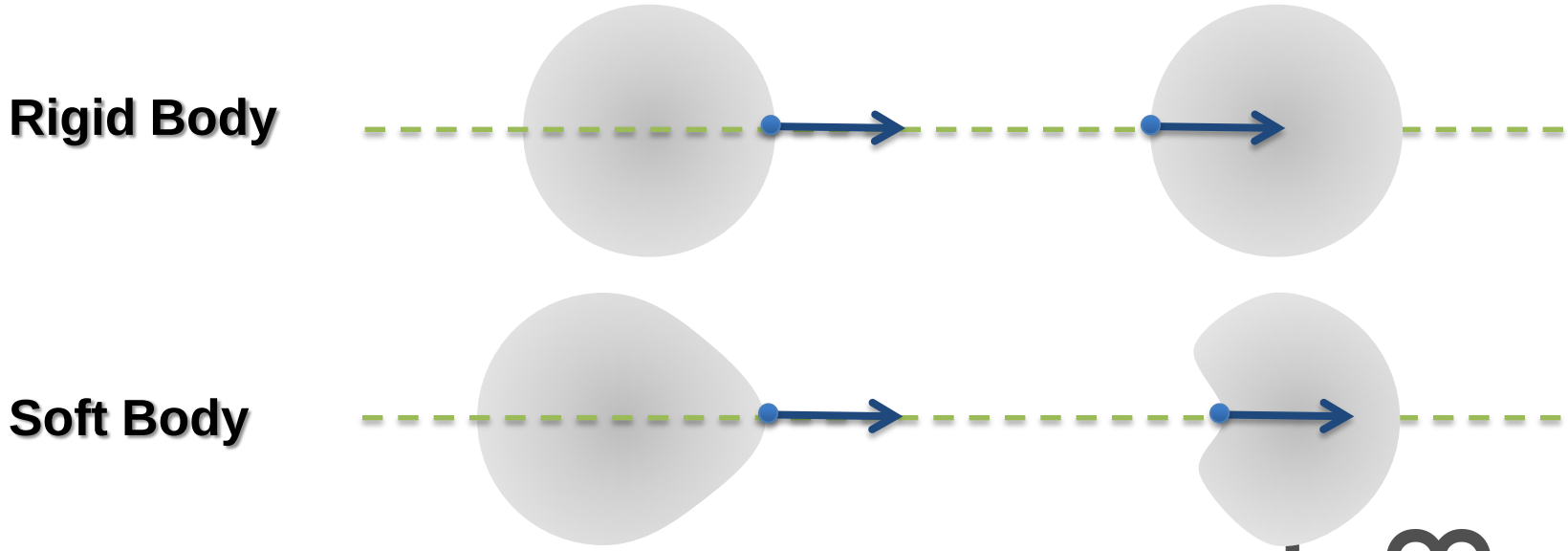


Rigid Bodies

Because of this rigidness for a rigid body only its position and orientation in space need to be stored.

They suffice to describe its instantaneous state.

Another special property of rigid bodies is that the point of action for a force is irrelevant – only the line of action matters.



Rigid Bodies – Particle Model

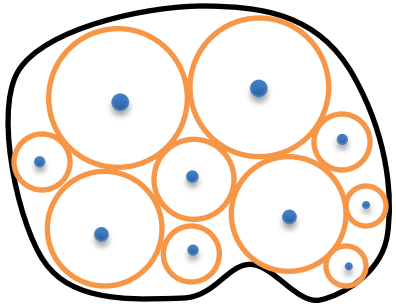
We already know particles. They have mass but no orientation. A group of particles, however, has orientation.

We can use a **set of particles** $\mathbf{B}$ to model a rigid body. Each particle has its own properties: mass, position, velocity, etc. But they are all linked together by their constant distance to each other. We will use this to deduce many important properties of rigid bodies.

We will see later that many properties of the whole rigid body are made up of the sum of its particles.

The mass of the rigid body for example is:

$$m = \sum_{i \in \mathbf{B}} m_i = \int_{\mathbf{B}} dm$$



Where We Are

Rigid Body Update Code

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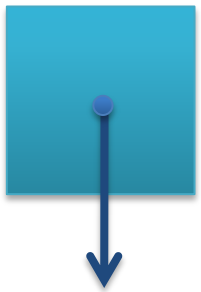
```

Field Forces

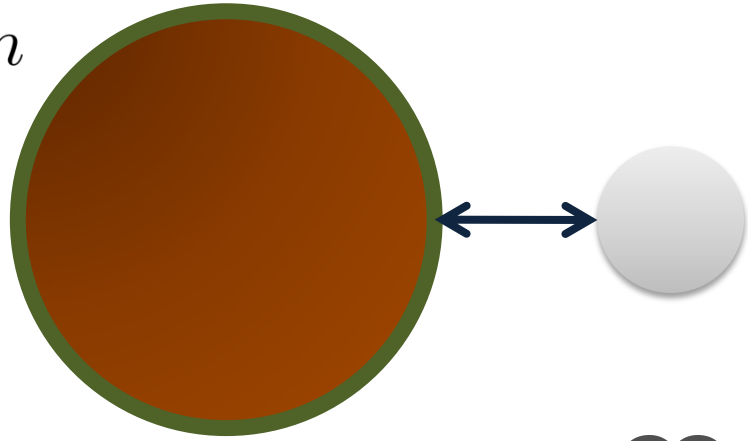
Definition

A **field force (Feldkraft)** is a force that is not transferred through direct contact but through a **force field (Kraftfeld)**.

Example: Gravity



$$F_g = 9.81 \frac{\mathrm{m}}{s^2} m$$



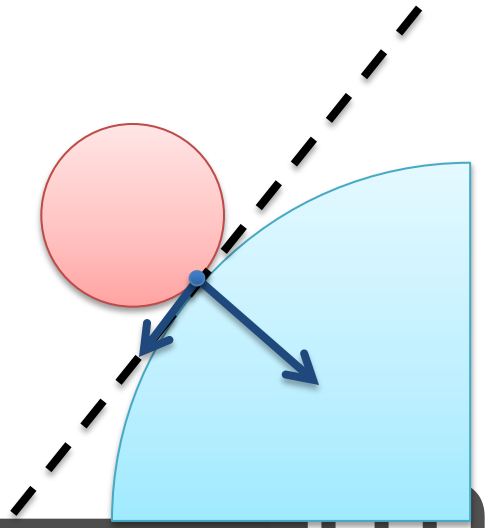
Contact Forces

Definition

A **contact force (Kontaktkraft)** is a force that is transmitted through a **contact point** of two bodies that touch.

Because of $actio = -reactio$ both bodies apply the same force (but in opposite directions) on each other.

It consists of a **normal force (Normalkraft)** perpendicular to the contact plane and **a friction force (Reibungskraft)** that lies on the contact plane.



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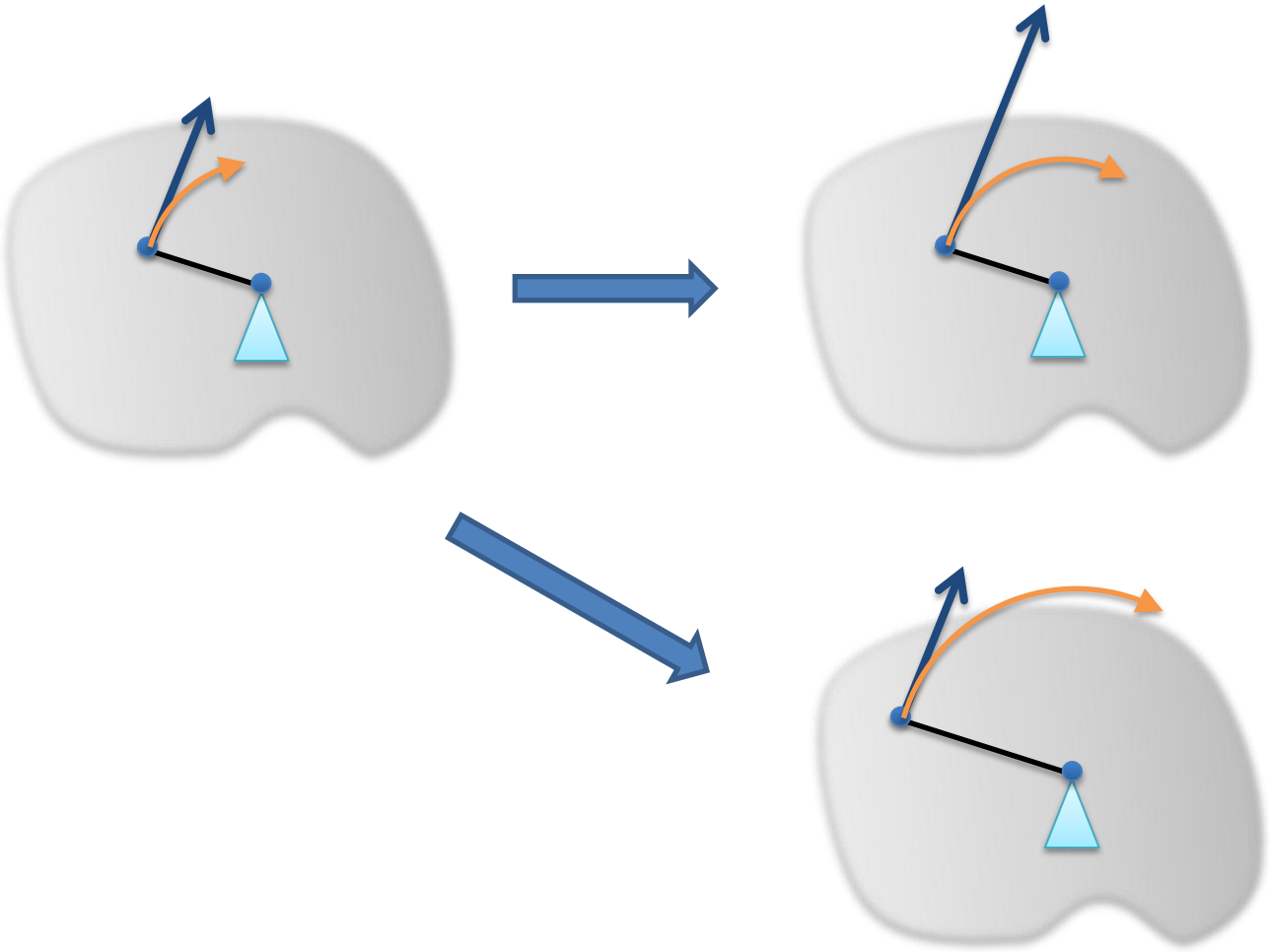
Moment (Motivation)

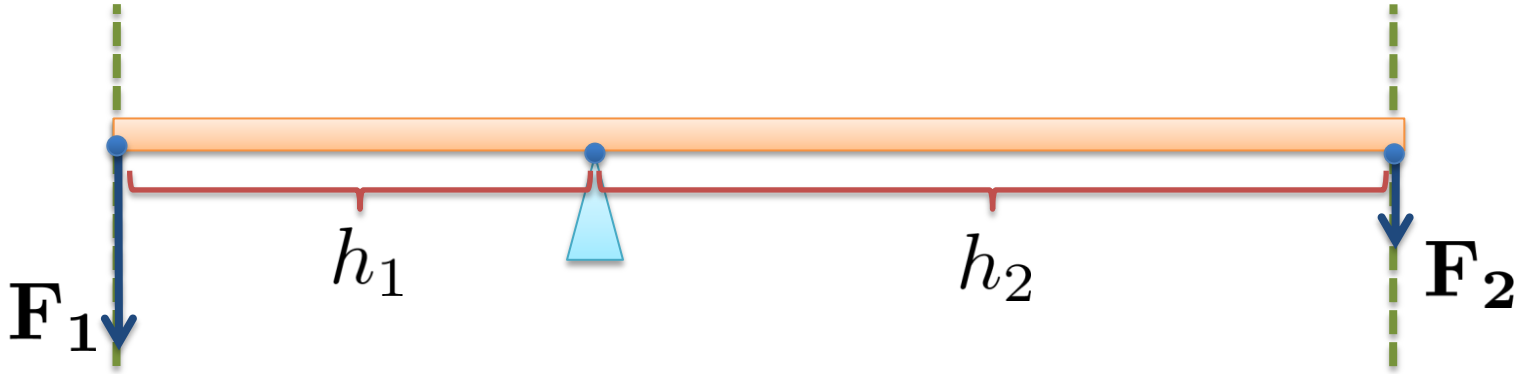


Law of the Lever

Law of the Lever (Hebelgesetz)

A level is in equilibirum if the forces applied on each end is inversely proportional to the distance of the points of action to the pivot:

$$\mathbf{F}_1 \, h_1 = \mathbf{F}_2 \, h_2$$

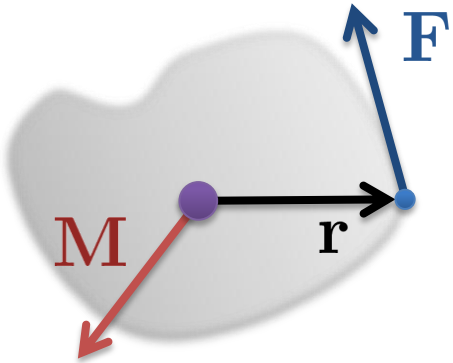


Moment

Definition

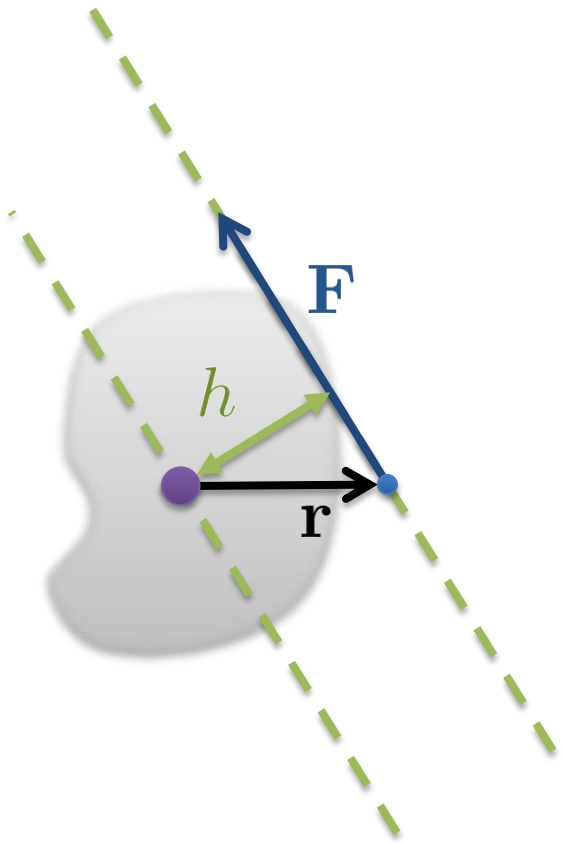
Moment (Drehmoment) is a measure for the ability of a force to rotate an object.
It is always defined with respect to a point of reference $\mathbf{r_x}$:

$$\mathbf{M}_{\mathbf{F}}^{(\mathbf{x})} = (\mathbf{r_F} - \mathbf{r_x}) \times \mathbf{F} = \mathbf{r}_{\mathbf{F}}^{(\mathbf{x})} \times \mathbf{F}$$



Moment

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$
$$M = h\,F$$



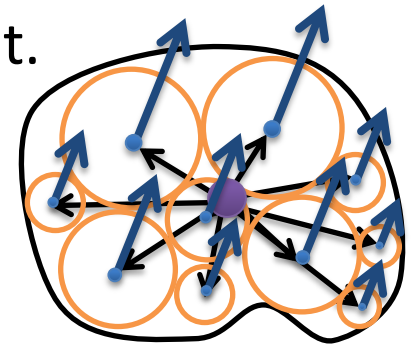
Net Force and Net Torque

If we have forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ (with points of action $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ etc), then we can calculate the net force and net moment in respect to a point of reference $\mathbf{x}$ with:

$$\mathbf{F} = \sum_i \mathbf{F}_i$$

$$\mathbf{M}^{(\mathbf{x})} = \sum_i \mathbf{M}_{\mathbf{F}_i}^{(\mathbf{x})}$$

That is, we can simply add up the forces and moments to get the net force and net moment.



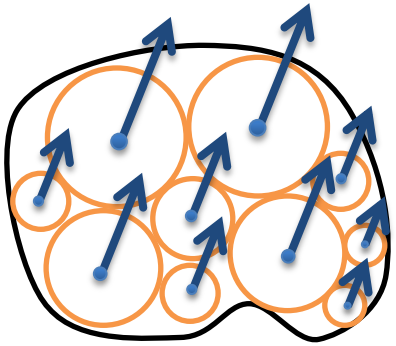
Center of Mass (Motivation)

What is a good choice for our point of reference?

It would be nice to choose it in such a way that applying a force to it won't induce any actual moment on an unconstrained rigid body, that is $\mathbf{M} \stackrel{!}{=} \mathbf{0}$.

This means that every particle as well as the whole rigid body will experience the same acceleration. So if a force $\mathbf{F}$ is applied to the point of reference, the whole rigid body should be accelerated by $\mathbf{a} = \frac{\mathbf{F}}{m}$. m is the mass of the whole rigid body, ie $m = \sum_i m_i$.
Then:

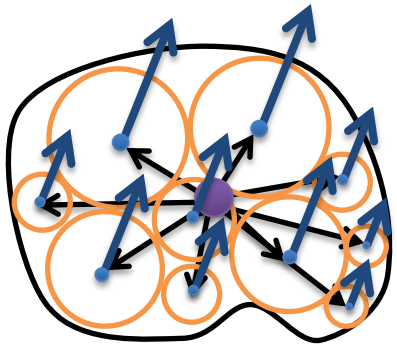
$$\mathbf{F}_i = m_i \, \mathbf{a} = \frac{m_i}{m} \, \mathbf{F}$$



Center of Mass (Deduction)

$$\sum_i \mathbf{M}_i \quad (\text{net moment}) = \mathbf{M} = 0$$

$$\sum_i (\mathbf{r_i} - \mathbf{r}) \times \mathbf{F}_i = 0$$



$$\sum_i (\mathbf{r_i} - \mathbf{r}) \times \frac{m_i}{m} \mathbf{F} = \left(\sum_i \frac{m_i}{m} (\mathbf{r_i} - \mathbf{r}) \right) \times \mathbf{F} = 0$$

Because the last statement has to hold for any force $\mathbf{F}$, it follows that $\sum_i \frac{m_i}{m} (\mathbf{r_i} - \mathbf{r}) = 0$, that is:

$$\sum_i \frac{m_i}{m} (\mathbf{r_i} - \mathbf{r}) = \sum_i \frac{m_i}{m} \mathbf{r_i} - \sum_i \frac{m_i}{m} \mathbf{r} = \sum_i \frac{m_i}{m} \mathbf{r_i} - \mathbf{r} = 0$$

$$\mathbf{r} = \sum_i \frac{m_i}{m} \mathbf{r_i} = \frac{1}{m} \int \mathbf{r} dm$$

Center of Mass

Definition

The **center of mass ((Masse-)Schwerpunkt)** $\mathbf{cm}$ of a rigid body is the mass-weighted average of all particle positions:

$$\mathbf{r}_{cm} = \sum_i \frac{m_i}{m} \mathbf{r_i} = \frac{1}{m} \int \mathbf{r} dm$$

The center of mass of a rigid body moves as if the whole mass of the rigid body were focused in it and all external forces acted on it.

Where We Are

Rigid Body Update Code

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}
```

Kinetics

Linear Momentum

Definition

The **linear momentum (Impuls)** $\mathbf{p}$ of a particle i is defined as:

$$\mathbf{p}_i = m\,\mathbf{v}_i$$

The linear momentum of a rigid body is the sum of the linear momentums of all its particles:

$$\mathbf{p} = \sum_i \mathbf{p}_i = \int \mathbf{v} dm = m\,\mathbf{v}_{cm}$$

The linear momentum of a system of bodies is a **conserved quantity (Erhaltungsgröße)**, ie the total amount never changes inside the system.

This is called **conservation of momentum (Impulserhaltung)**.

Linear Momentum and Force

$$\mathbf{F} = m\,\mathbf{a}$$

$$\mathbf{p} = m\,\mathbf{v}$$

Look at the derivative of $\mathbf{p}$:

$$\dot{\mathbf{p}} = \dot{m}\,\mathbf{v} + m\,\dot{\mathbf{v}}$$

But m is constant and thus $\dot{m} = 0$.

We get:

Derivative of Linear Momentum = Force

The linear momentum changes according to the applied force:

$$\dot{\mathbf{p}} = m\,\dot{\mathbf{v}} = m\,\mathbf{a} = F$$

Newton's First Law of Motion

Newton's First Law

Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it. This is the concept of **inertia (Trägheit).**

Proof:

This follows directly from the conservation of momentum:

If $\mathbf{F} = 0$ then $\dot{\mathbf{p}} = 0$, and then $\mathbf{p} = m\,\mathbf{v} = const$ and since the mass m is constant, we see that the velocity is constant, too.

Angular Momentum

Definition

The **angular momentum (Drehimpuls)** $\mathbf{L}$ of a particle i is defined as:

$$\mathbf{L}_i^{(\mathbf{x})} = (\mathbf{r}_i - \mathbf{r}_x) \times \mathbf{p}_i = \mathbf{r}_i^{(x)} \times \mathbf{p}_i$$

with respect to a point of reference $\mathbf{x}$.

The angular momentum of a rigid body is the sum of the angular momentum of its particles:

$$\mathbf{L}^{(x)} = \sum_i \mathbf{L}_i^{(\mathbf{x})} = \sum_i \mathbf{r}_i^{(x)} \times \mathbf{p}_i = \int \mathbf{r}^{(x)} \times \mathbf{v_i} dm$$

Angular momentum is a **conserved quantity**, too, that is it stays constant for a system of particles (except if outer forces are applied).

Angular Momentum and Moment

Derivative of Angular Momentum = Moment

Angular momentum changes according to the applied momentum (around the center of mass):

$$\dot{\mathbf{L}}^{(cm)} = \mathbf{M}^{(cm)}$$

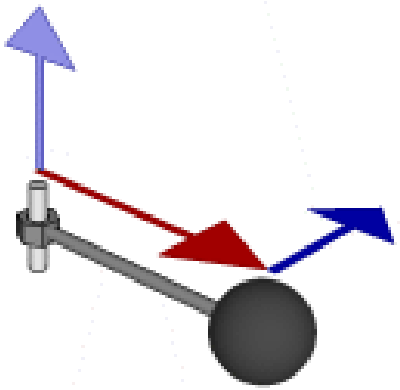
Proof Sketch:

$$\dot{L}_{\mathbf{x}}^{(cm)} = \frac{d}{dt}\left(\mathbf{r}_{\mathbf{x}}^{(cm)} \times \mathbf{p}_{\mathbf{x}}\right)$$

$$\dot{L}_{\mathbf{x}}^{(cm)} = \underbrace{\dot{\mathbf{r}}_{\mathbf{x}}^{(cm)} \times \mathbf{p}_{\mathbf{x}}}_{=\mathbf{0}} + \underbrace{\mathbf{r}_{\mathbf{x}}^{(cm)} \times \dot{\mathbf{p}}_{\mathbf{x}}}_{\mathbf{M}_{\mathbf{x}}^{(cm)}} = \mathbf{M}_{\mathbf{x}}^{(cm)}$$

$\tau = \mathbf{r} \times \mathbf{F}$

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$



Integration

The problem is that you know the net moment and net torque, but want to know the linear and angular momentum.

From the differential equation we need to evaluate:

$$\mathbf{p}\left(t+dt\right) = \int_{t}^{t+dt} \mathbf{F}dt + \mathbf{p}\left(t\right)$$

$$\mathbf{L}\left(t+dt\right) = \int_{t}^{t+dt} \mathbf{M}dt + \mathbf{L}\left(t\right)$$

The integration steps in a rigid body simulation can be as difficult and complicated as you want.

To solve this you can eg use a simple Euler step:

$$\mathbf{p}(t+\Delta t) = \mathbf{p}(t) + \mathbf{F}(t)\,\Delta t$$

$$\mathbf{L}(t+\Delta t) = \mathbf{L}(t) + \mathbf{M}(t)\,\Delta t$$

Summary

- Linear Momentum $\mathbf{p} = m\,\mathbf{v}$
- Angular Momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Conservation of Linear Momentum
- Conservation of Angular Momentum
- Linear Momentum = Integrated Force: $\dot{\mathbf{p}} = \mathbf{F}$
- Angular Momentum = Integrated Moment: $\dot{\mathbf{L}} = \mathbf{M}$
- Integration = actual problem

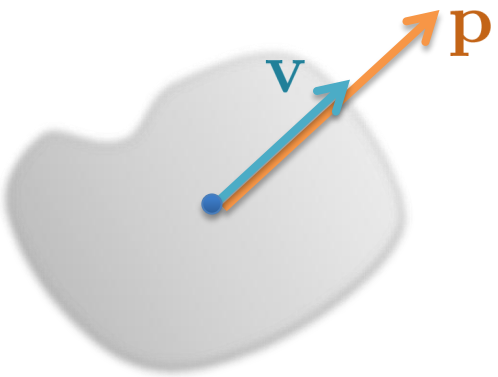
Where We Are

Rigid Body Update Code

```
for each (RigidBody* body in scene->rigidBodies) {
   auto contacts = scene->determineContacts(body);
   auto fieldForces = scene->determineFieldForces(body);
   auto totalForce = computeTotalForce(contacts, fieldForces);
   auto totalMoment = computeTotalMoment(contacts);
   body->linearMomentum.integrate(timeStep, totalForce);
   body->angularMomentum.integrate(timeStep, totalMoment);
   body->velocity = body->linearMomentum / body->mass;
   body->angularVelocity = body->invInertiaTensor *
                           body->angularMomentum;
   body->position.integrate(timeStep, body->velocity);
   body->rotation.integrate(timeStep, body->angularVelocity);
}
```

Linear Momentum and Linear Velocity

From the definition of linear momentum $\mathbf{p} = m\,\mathbf{v}$ we can directly identify $\mathbf{v} = \frac{\mathbf{p}}{m}$.



Angular Momentum and Angular Velocity

We know:

$$\mathbf{L}^{(cm)} = \sum_i \mathbf{r}_i^{(cm)} \times \mathbf{p_i} = \sum_i \mathbf{r}_i^{(cm)} \times m_i \, \mathbf{v}_i \, .$$

$\mathbf{v}_i$ depends on the angular velocity ω indirectly.

Using this fact, it is possible to transform the equation to

$$\mathbf{L}^{(cm)} = \Phi \, \omega,$$

where Φ is the **inertia tensor (Trägheitstensor)**.

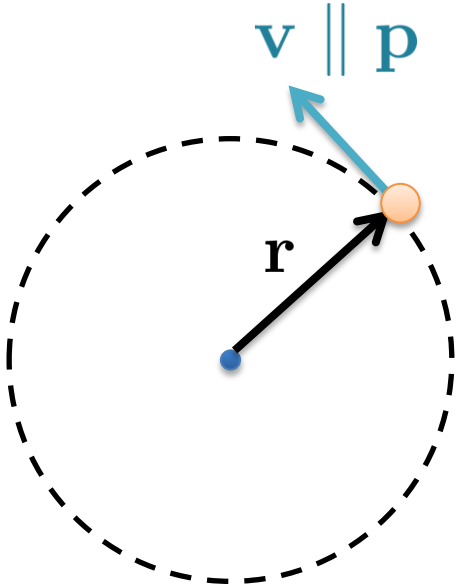
Inertia Tensor in 2D

In 2D the moment definition can be simplified to scalars (like we did for the moment):

$$L_i^{(cm)} = h_i^{(cm)}\, p_i \text{ with } h_i^{(cm)} = r_i^{(cm)} \sin\alpha$$

If we look at an angular motion, it can also be thought of as circular motion around the center of gravity and since $\mathbf{p_i}$ is collinear with $\mathbf{v}$, we know that it is perpendicular to $\mathbf{r_i}$, this gives $h_i = r_i$. From the slide about circular movement we also know that

$$\mathbf{v_i} = r_i\,\omega \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow v_i = r_i\,\omega \,.$$



Inertia Tensor in 2D

$$L_i^{(cm)} = h_i^{(cm)}\, p_i \text{ with } h_i^{(cm)} = r_i^{(cm)} \sin\alpha$$

$$\mathbf{v_i} = r_i\,\omega \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow v_i = r_i\,\omega$$

$$L^{(cm)} = \sum_i L_i^{(cm)} = \sum_i r_i\, p_i = \sum_i r_i\, m_i\, v_i = \sum_i m_i\, r_i\ (r_i\,\omega)$$

$$= \sum_i m_i\, r_i^2\,\omega = \left(\sum_i m_i\, r_i^2\right)\ \omega = \Phi\,\omega$$

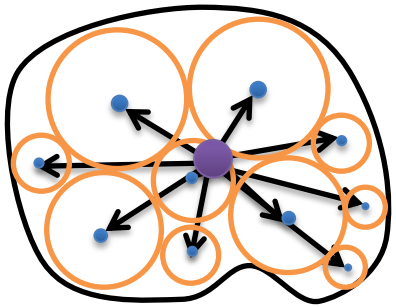
$$\Phi = \sum_i m_i\, r_i^2 = \int r^2 dm$$

Inertia Tensor in 2D

Definition

In 2D the inertia tensor is just a scalar. It is the squared distance of all particles to the center of gravity weighted by their mass:

$$\Phi = \sum_i m_i \, r_i^{(cm)^2} = \int r^{(cm)^2} dm$$



Inertia Tensor in 3D

Definition

In 3D the inertia tensor is a 3×3 matrix:

$$\Phi = \sum_i m_i \begin{pmatrix} r_{iy}^2 + r_{iz}^2 & -r_{ix}\, r_{iy} & -r_{ix}\, r_{iz} \\ -r_{ix}\, r_{iy} & r_{ix}^2 + r_{iz}^2 & -r_{iy}\, r_{iz} \\ -r_{ix}\, r_{iz} & -r_{iy}\, r_{iz} & r_{ix}^2 + r_{iy}^2 \end{pmatrix}$$

The deduction is similar to the 2D case, only more heavy-handed.

Intuition behind the Inertia Tensor

It is possible to rotate many objects in such a way that Φ is diagonal.

$$\Phi = \begin{pmatrix} \Phi_x & 0 & 0 \\ 0 & \Phi_y & 0 \\ 0 & 0 & \Phi_z \end{pmatrix}$$

$$\mathbf{L} = \Phi\,\boldsymbol{\omega} \Leftrightarrow \boldsymbol{\omega} = \Phi^{-1}\,\mathbf{L}$$

The bigger the diagonal entry the more inert the object is regarding a rotation around that axis.

The smaller Φ is in total, the faster the object's rotation axis can change.

Where We Are

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   auto fieldForces = scene->determineFieldForces(body);
   auto totalForce = computeTotalForce(contacts, fieldForces);
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   body->linearMomentum.integrate(timeStep, totalForce);
   body->angularMomentum.integrate(timeStep, totalMoment);
   body->velocity = body->linearMomentum / body->mass;
   body->angularVelocity = body->invInertiaTensor *
                           body->angularMomentum;
   body->position.integrate(timeStep, body->velocity);
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}
```

Where We Are

Rigid Body Update Code

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Questions?

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- “Mathematics for 3D Game Programming & Computer Graphics” by Eric Lengyel
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